3. Failure Theories

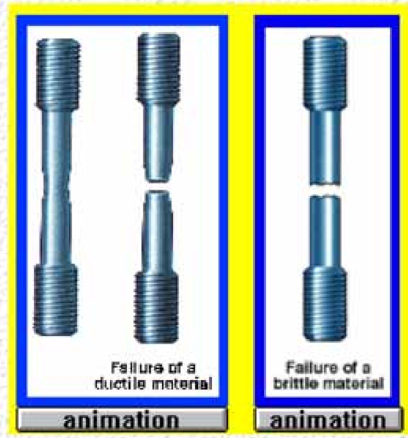
3.1 Definitions of Failure
3.2 Types of Failure
- Yielding, Fracture, Buckling
3.3 Factors Affecting Mode of Failure
3.4 Uniaxial Versus Multiaxial (Combined Loading) Cases
3.5 Yielding and Fracture Criteria for Multiaxial Stress State
3.6 Shaft Subjected to Combined Axial Force and Twisting Moment
3.7 Application of Failure Theories to Design

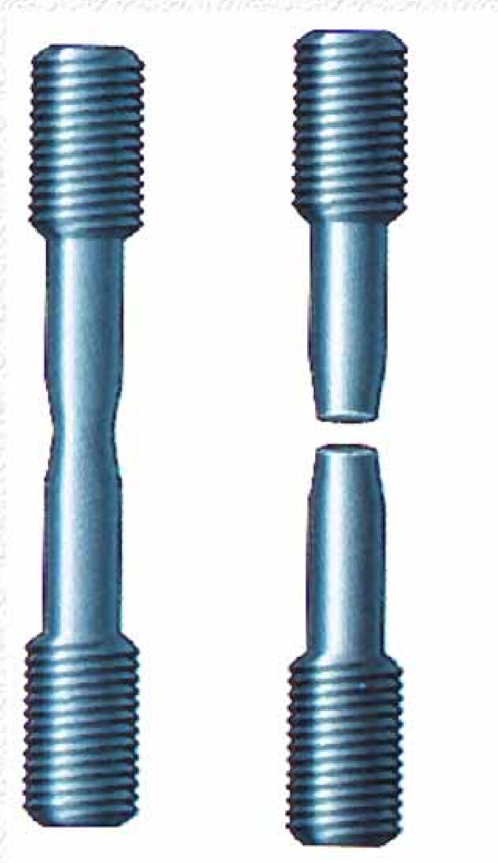
Failure Theories

Definitions of Failure

Definition

Failure is any action leading to an inability of the structure (or its components) to function in the manner intended.





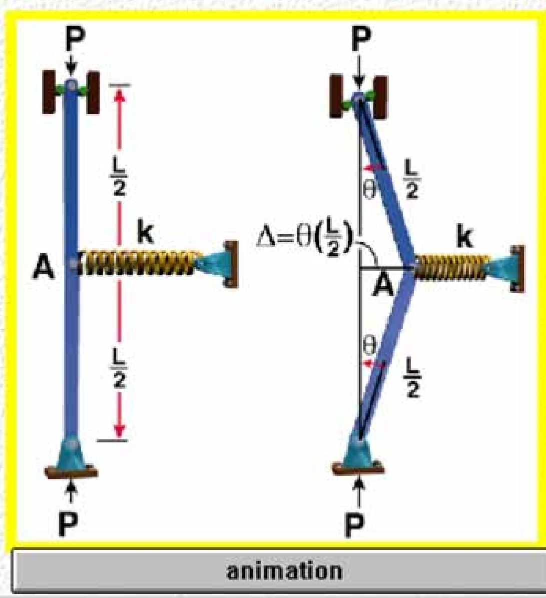
Failure of a Ductile Material

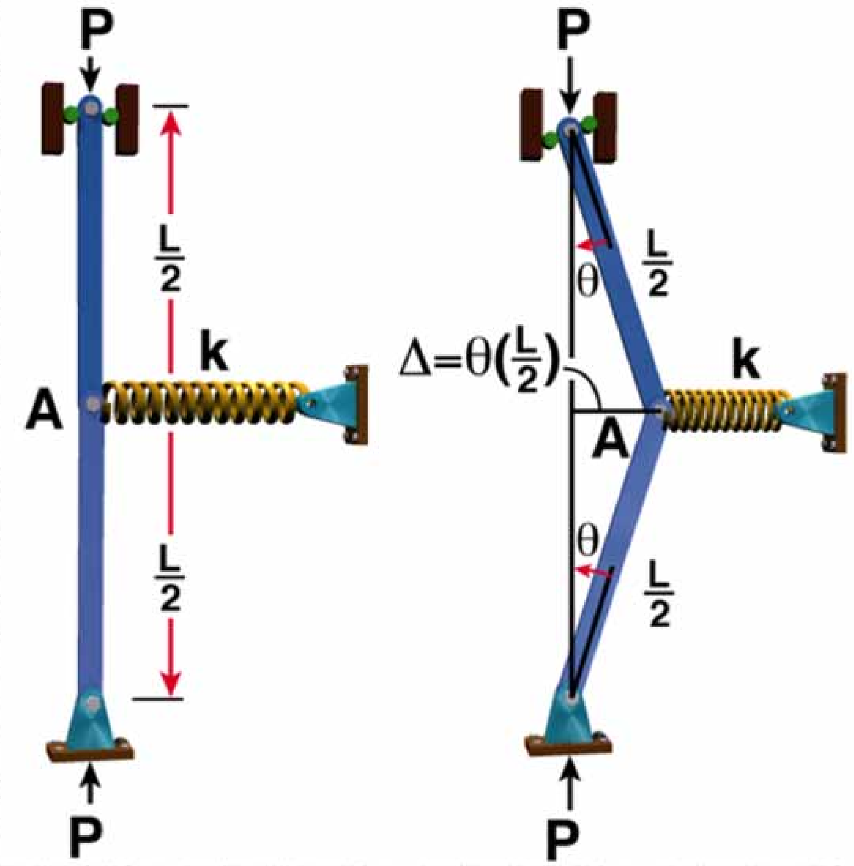
Failure of a
brittle material

Types of Failure

Types (or Modes) of Failure

- Excessive elastic deformations
- Permanent deformation (or yielding)
- Fracture
  - Brittle (e.g., concrete, stone, glass)
  - Ductile (e.g., mild steel, aluminum,copper)
  - Progressive - fatigue
- Instability





Factors Affecting Mode of Failure

Characteristics of structural member
- material
- geometry and shape

Loading
- loading configuration and rate
- surrounding media

Uniaxial versus Multiaxial (combined loading) cases

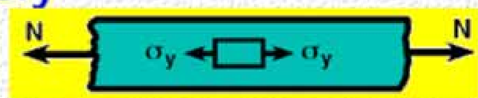
- For uniaxial stress (and loading) the onset of failure (by yielding or fracture) can be predicted from the stress-strain diagram.
- For multiaxial (combined loading) case, a response quantity (stress, strain or energy) associated with failure is chosen.
  - A maximum (or critical) value of the quantity is selected to predict the onset of failure
  - A uniaxial (or torsion) test is used to determine the maximum (or critical) value.

3.5 Yielding and Fracture Criteria for Multiaxial Stress State

- 3.5.1 Maximum Principal Stress Theory
- 3.5.2 Maximum Strain Theory
- 3.5.3 Maximum Shear Stress Theory
- 3.5.4 Maximum Total Strain Energy Theory
- 3.5.5 Maximum Energy of Distortion Theory
- 3.5.6 Maximum Octahedral Shear Stress Theory

Maximum Principal Sress Theory

Maximum Principal Stress Theory (Rankin's Theory)

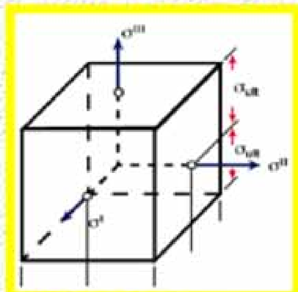


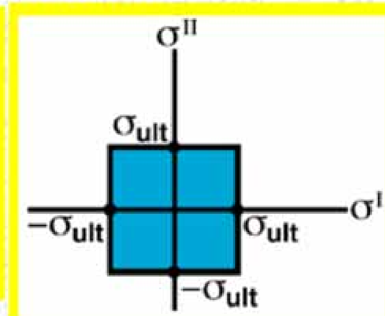
- Fracture is assumed to occur at a point when the maximum principal stress at that point reaches the ultimate stress in simple tension or compression, for the material.

$$\boxed{-\sigma_{ult.} \le \sigma^{I} \le \sigma_{ult.}}$$

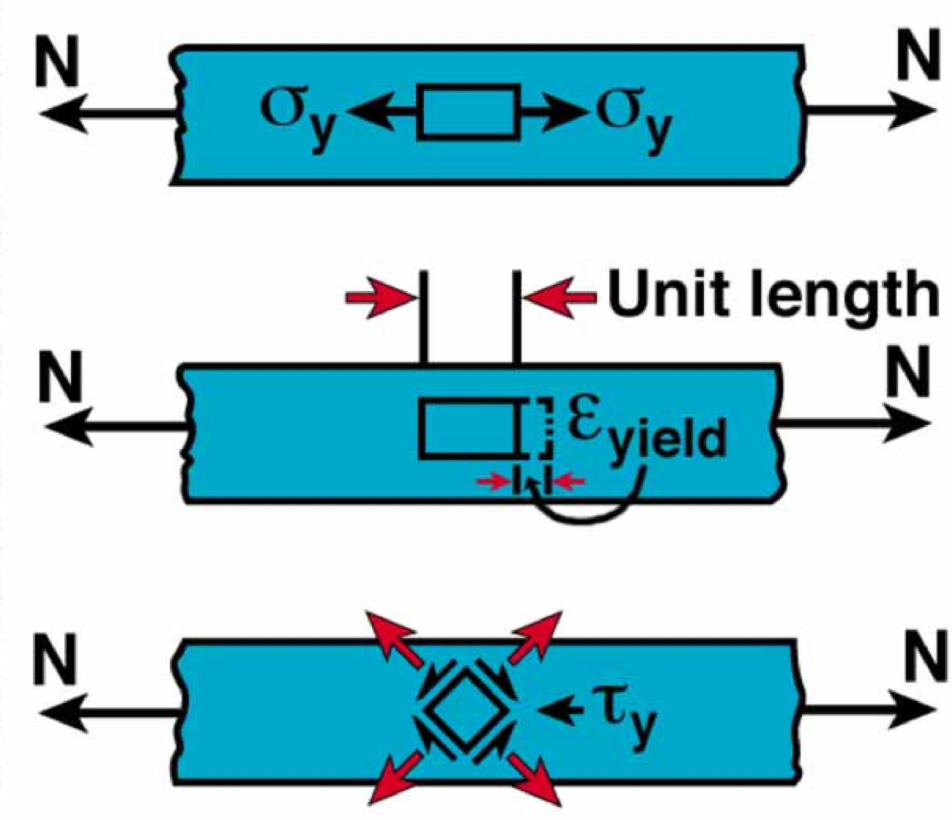
$$\boxed{-\sigma_{ult.} \le \sigma^{II} \le \sigma_{ult.}}$$

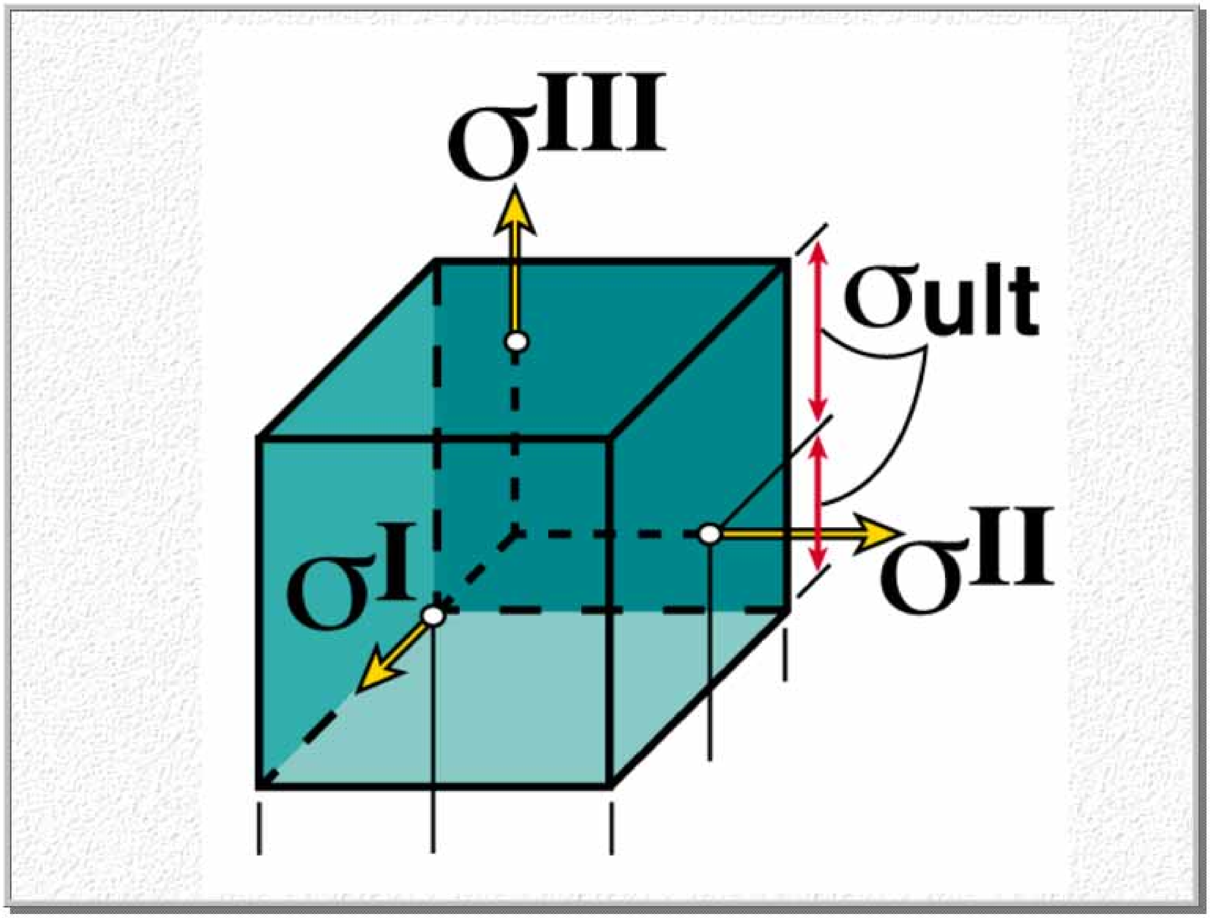
$$\boxed{-\sigma_{ult.} \le \sigma^{III} \le \sigma_{ult.}}$$



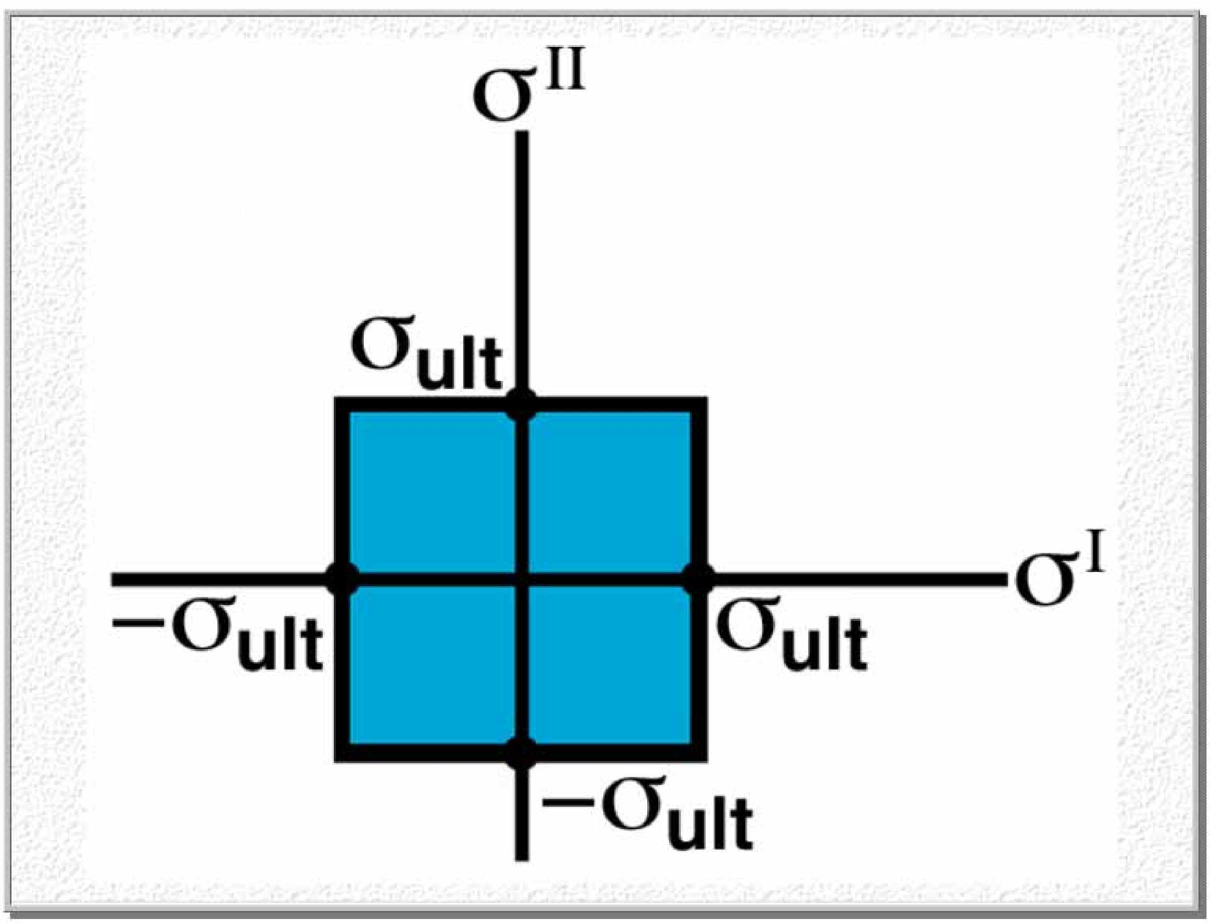


- The criterion is suitable for brittle materials.





σIII
σult
σI
σII



σII
σult
σI
−σult
σult
−σult

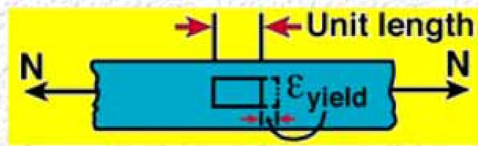
Maximum Strain Theory

(St. Venant's Theory)

- Yielding is assumed to begin at a point when the maximum strain at that point reaches a value equal to that of the yield strain in a simple tension test.
- In terms of principal strains

$$\varepsilon_{max} = \varepsilon^{I} = \frac{1}{E}\left(\sigma^{I} - \upsilon\sigma^{II} - \upsilon\sigma^{III}\right)$$

- In uniaxial tension



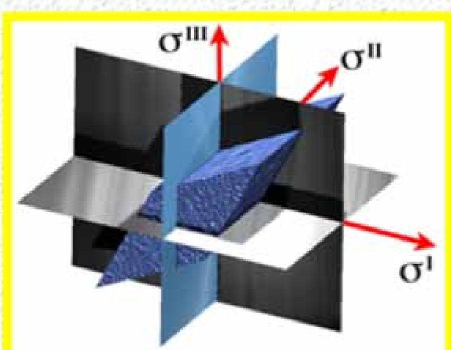
$$\varepsilon_{yield} = \frac{1}{E}\,\sigma_{yield}$$

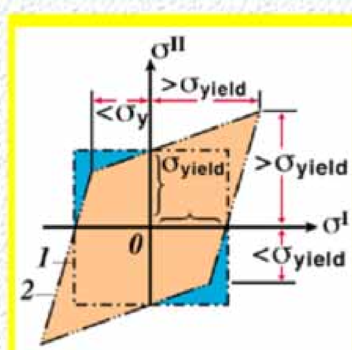
Maximum Strain Theory

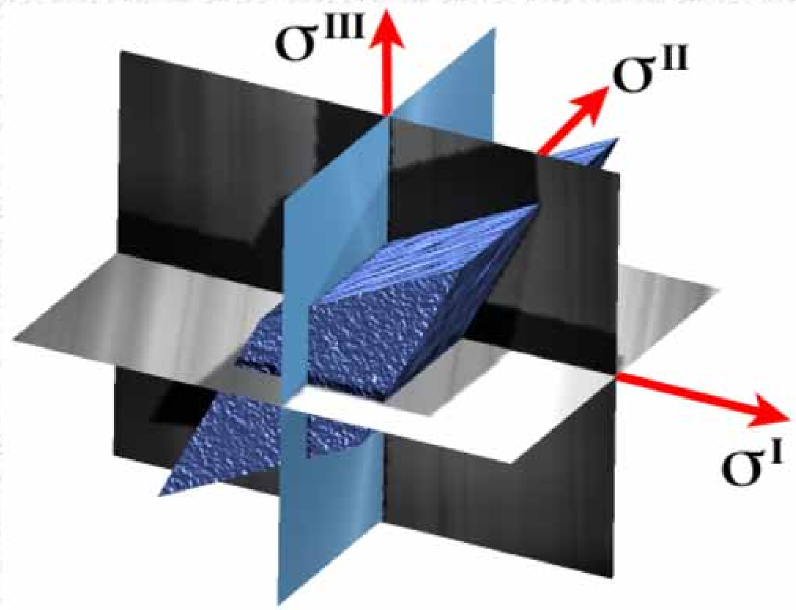
- Two-dimensional stress state $\sigma^{III} = 0$

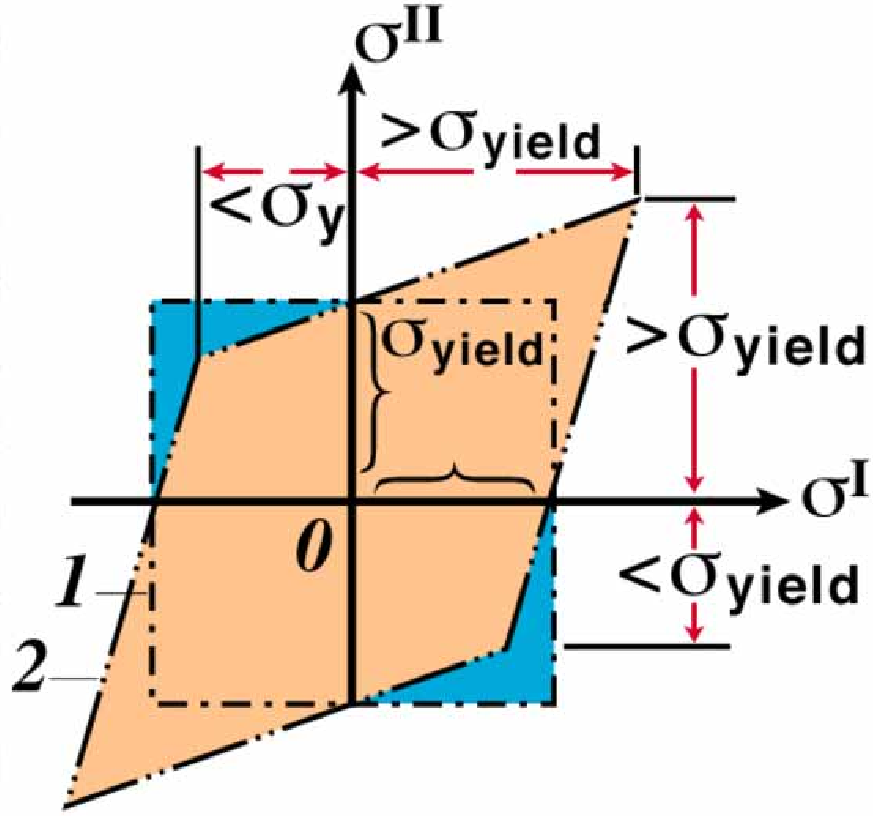
$$-\sigma_{yield} \le \sigma^{I} - \nu\,\sigma^{II} \le \sigma_{yield}$$

$$-\sigma_{yield} \le \sigma^{II} - \nu\,\sigma^{I} \le \sigma_{yield}$$






σIII
σII
σI


σII
>σyield
<σy
σyield
>σyield
σI
0
<σyield
1
2

Maximum Strain Theory

$$-\sigma_{yield} \le \sigma^{I} - \nu\,\sigma^{II} \le \sigma_{yield}$$

$$-\sigma_{yield} \le \sigma^{II} - \nu\,\sigma^{I} \le \sigma_{yield}$$

for

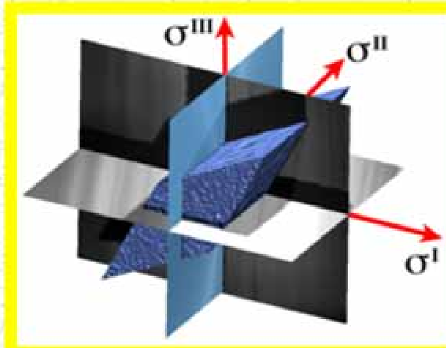
$$\sigma^{I} - \nu\,\sigma^{II} = \sigma_{yield}$$

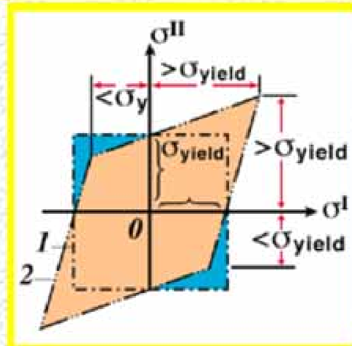
$$\sigma^{I} = \sigma_{yield} + \nu\,\sigma^{II}$$

for

$$\sigma^{I} - \nu\,\sigma^{II} = -\,\sigma_{yield}$$

$$\sigma^{I} = -\,\sigma_{yield} + \nu\,\sigma^{II}$$





Maximum Strain Theory

for

$$\sigma^{I} - \nu\,\sigma^{II} = \sigma_{yield}$$

$$\sigma^{I} = \sigma_{yield} + \nu\,\sigma^{II}$$

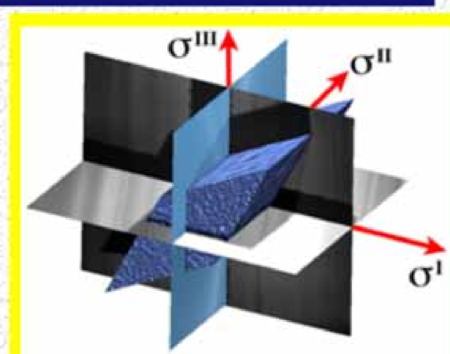
for

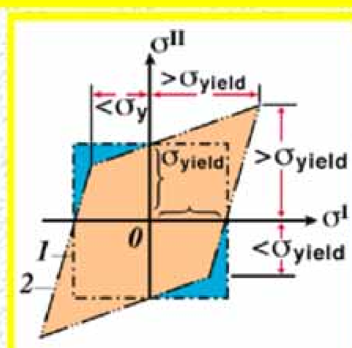
$$\sigma^{I} - \nu\,\sigma^{II} = -\,\sigma_{yield}$$

$$\sigma^{I} = -\,\sigma_{yield} + \nu\,\sigma^{II}$$

with similar equations for

$$\sigma^{II} - \nu\,\sigma^{I} = \pm\,\sigma\ \text{yield}$$



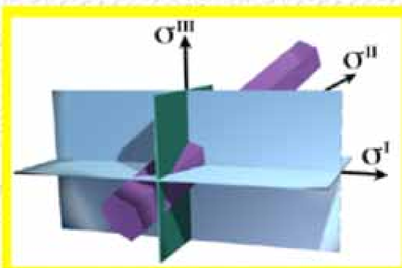


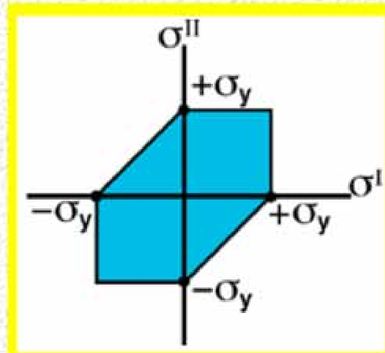
Maximum Shear Stress Theory

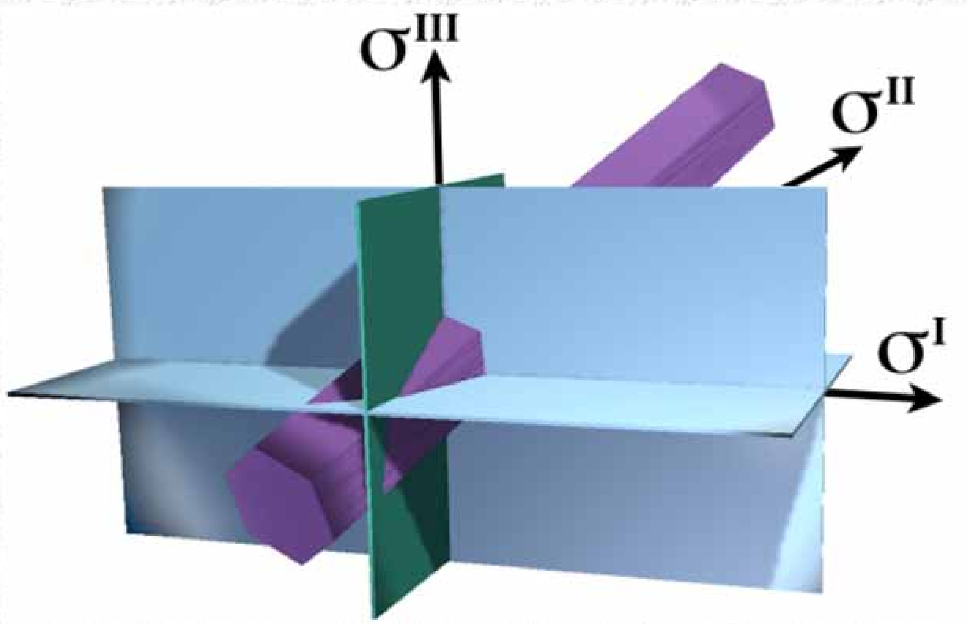
(Coulomb's, Tresca's or Guest's Theory)

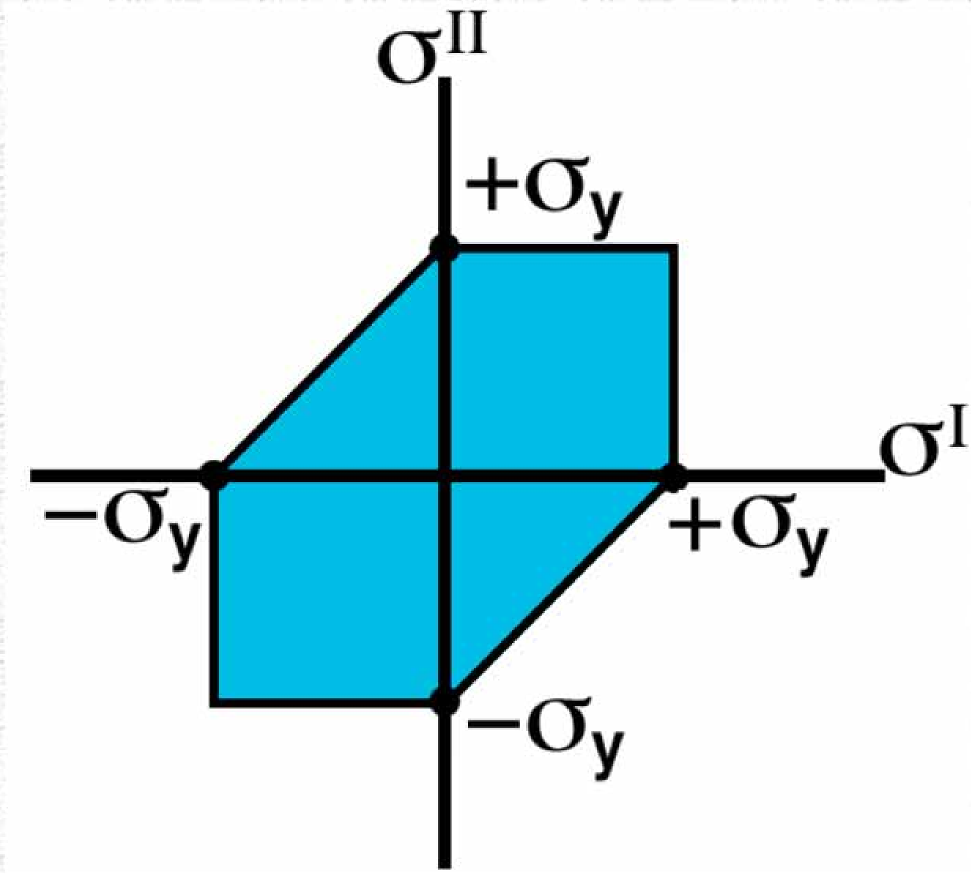
- Yielding is assumed to begin at a point when the maximum shear stress at that point equals the shear stress at the yield point in a simple tension test.
- In terms of principal stresses

$$\tau_{max} = \tau_{III,I} = \frac{1}{2}\,|\sigma^{III} - \sigma^{I}|$$









Maximum Shear Stress Theory

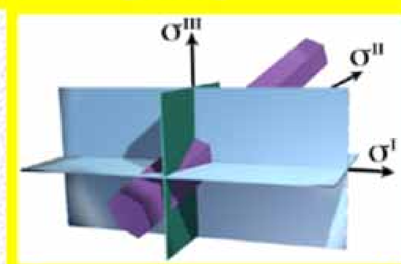
- In uniaxial tension

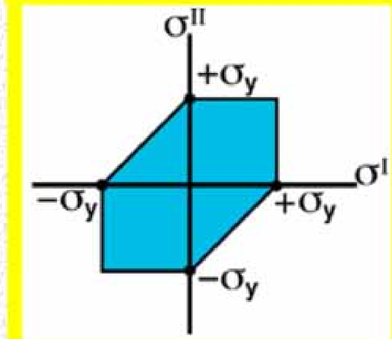
$$\boxed{\begin{aligned}&\sigma^{II} = \sigma^{III} = 0\\ &\tau_{max} = \frac{1}{2}\sigma_{yield}\end{aligned}}$$

- At the onset of yielding

$$\boxed{|\sigma^{I} - \sigma^{III}| = \sigma_{yield}}$$

- The criterion is suitable for ductile materials.





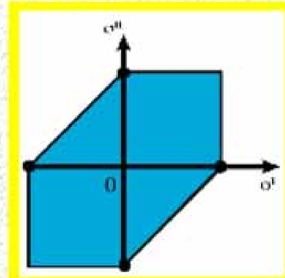
Maximum Shear Stress Theory

- Two-dimensional stress state $\sigma^{III} = 0$
  - if σ^{I} and σ^{II} have the same sign, then

$$\boxed{\tau_{max} = \frac{1}{2}\left(\sigma^{I} - \sigma^{III}\right)}$$

or

$$\boxed{\tau_{max} = \frac{1}{2}\left(\sigma^{II} - \sigma^{III}\right)}$$

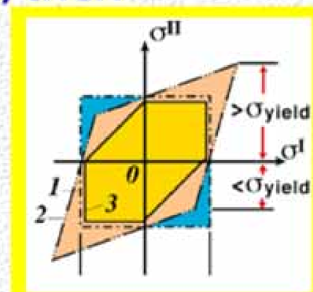


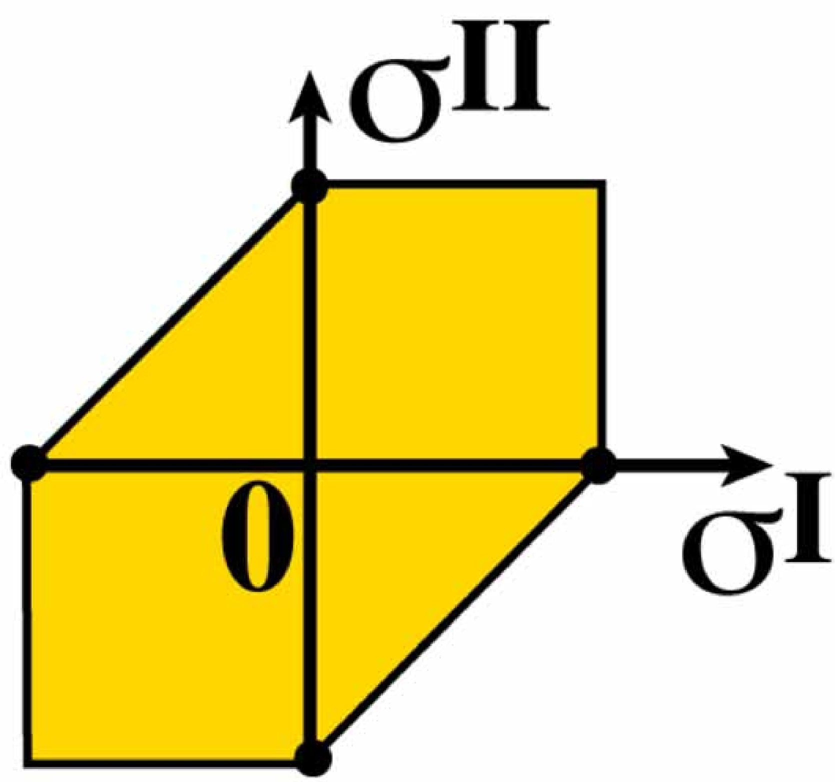
  - if σ^{I} and σ^{II} have opposite signs, then

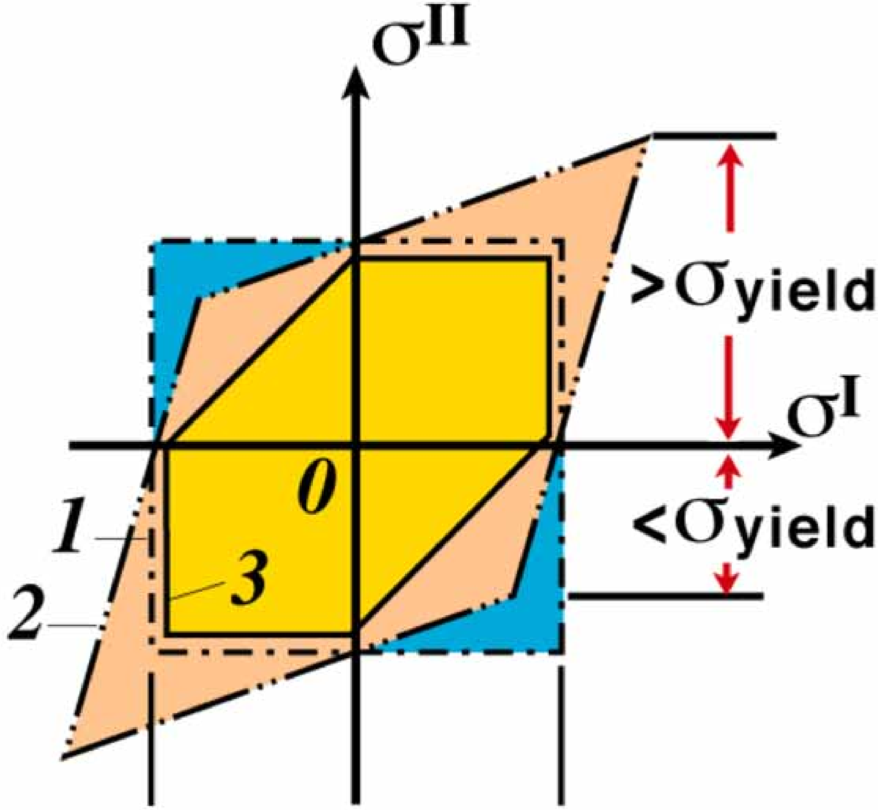
$$\boxed{\tau_{max} = \frac{1}{2}\left(\sigma^{I} - \sigma^{II}\right)}$$

or

$$\boxed{\tau_{max} = \frac{1}{2}\left(\sigma^{II} - \sigma^{I}\right)}$$







Maximum Total Strain Energy Theory

(Beltrami and Haigh's Theory)

- Yielding is assumed to begin at a point when the total strain energy density at that point reaches a value equal to the strain energy density in uniaxial tension (or compression).
- In terms of principal stresses, the total strain energy density

$$U = \frac{1}{2E}\left[(\sigma^{I})^2 + (\sigma^{II})^2 + (\sigma^{III})^2 - 2\upsilon(\sigma^{I}\sigma^{II} + \sigma^{II}\sigma^{III} + \sigma^{III}\sigma^{I}) \right]$$

- In uniaxial tension

Maximum Total Strain Energy Theory

- In terms of principal stresses, the total strain energy density

$$\boxed{U = \frac{1}{2E}\left[(\sigma^{I})^2 + (\sigma^{II})^2 + (\sigma^{III})^2 - 2\upsilon\left(\sigma^{I}\sigma^{II} + \sigma^{II}\sigma^{III} + \sigma^{III}\sigma^{I}\right)\right]}$$

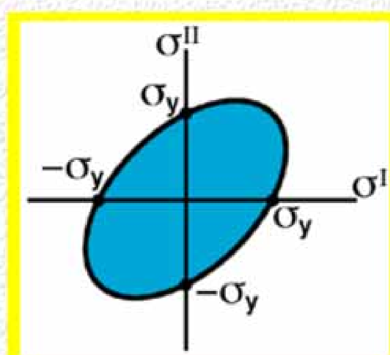
- In uniaxial tension

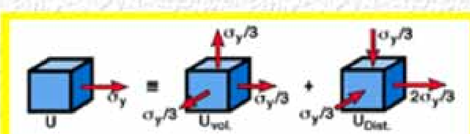
$$\boxed{U = \frac{1}{2E}\left(\sigma_{yield}\right)^2}$$

Maximum Energy of Distortion Theory

(Von Mises, Huber, Hencky Theory)

- Yielding is assumed to begin at a point when the distortional energy density (strain energy density associated with the change in shape - in terms of the deviatoric stresses and strains) at that point reaches a value equal to the distortional strain energy at yield in uniaxial tension (or compression).
- In terms of principal stresses





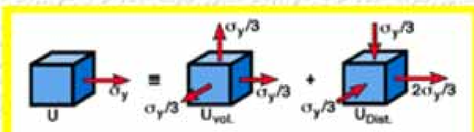
Maximum Energy of Distortion Theory

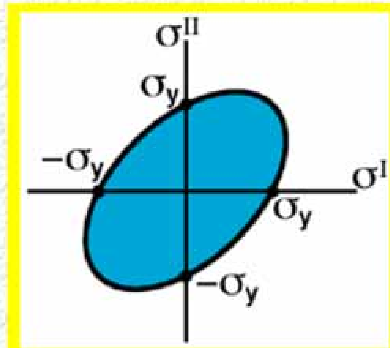
- In terms of principal stresses

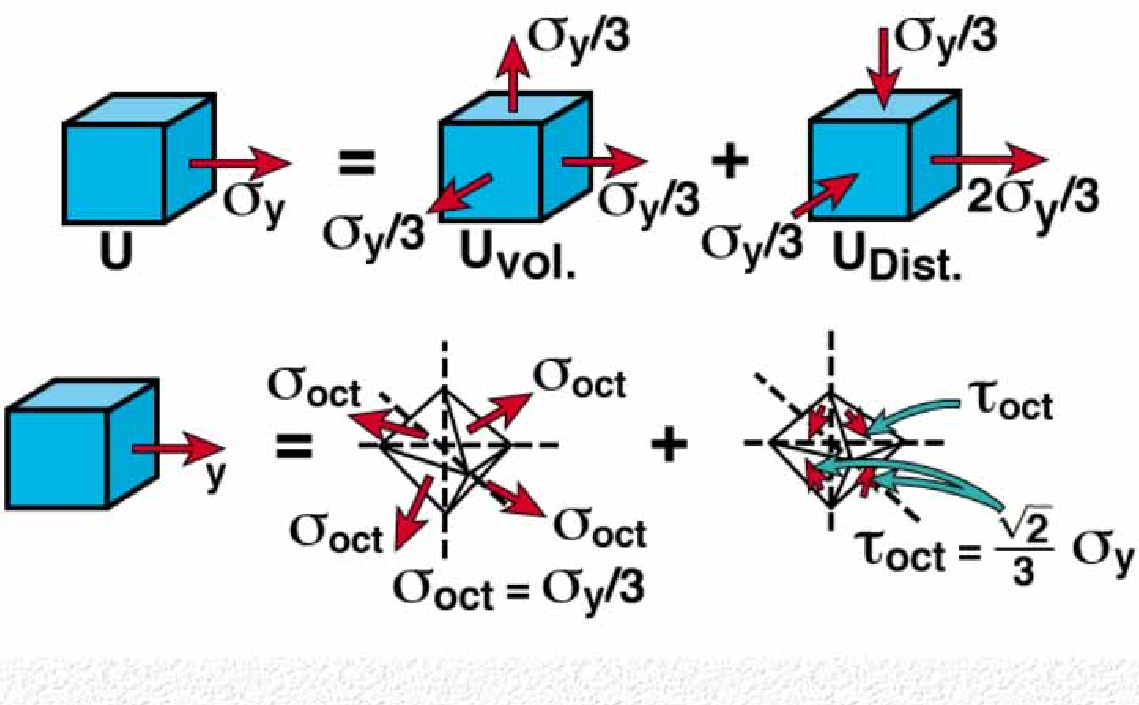
$$U_{dist.} = \frac{1}{12G}\left[(\sigma^{I} - \sigma^{II})^2 + (\sigma^{II} - \sigma^{III})^2 + (\sigma^{III} - \sigma^{I})^2\right]$$

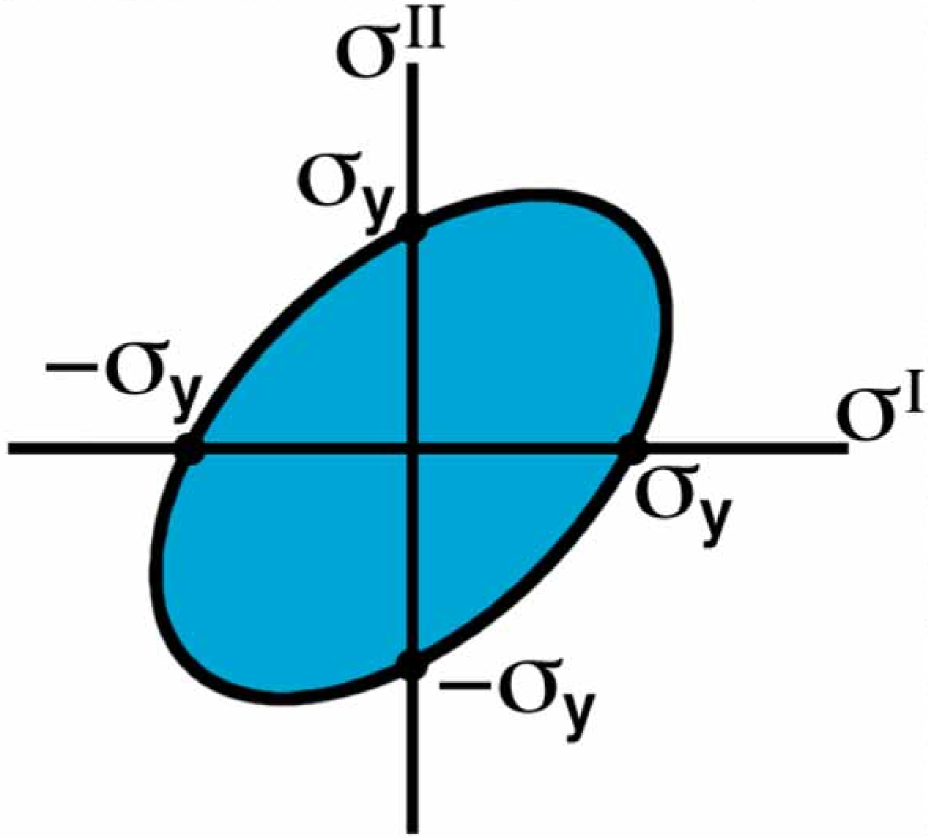
- Yielding begins when

$$U_{dist.} = \frac{1}{6G}(\sigma_{yield})^2$$









Maximum Energy of Distortion Theory

- Two-dimensional state of stress, $\sigma^{III} = 0$

$$\left(\sigma^{I}\right)^2 - \sigma^{I}\sigma^{II} + \left(\sigma^{II}\right)^2 = \left(\sigma_{yield}\right)^2$$

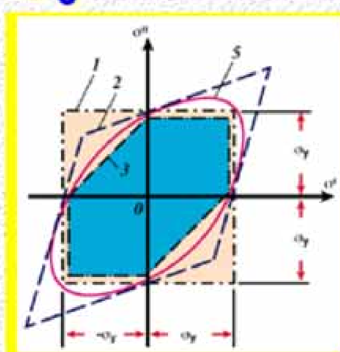
or

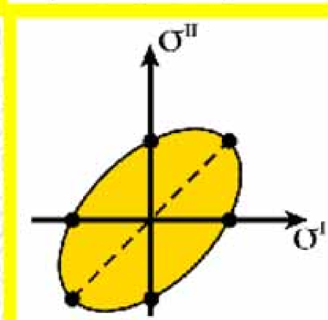
$$\left(\sigma_e\right)^2 = \left(\sigma_{yield}\right)^2$$

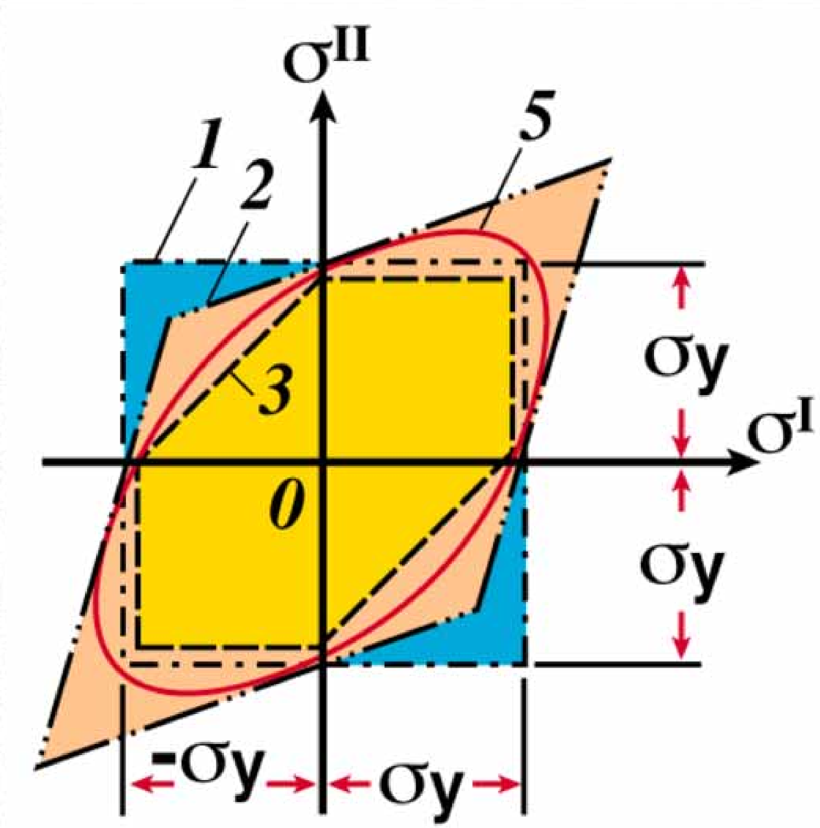
where

$$\sigma_e = \left[\left(\sigma^{I}\right)^2 - \sigma^{I}\sigma^{II} + \left(\sigma^{II}\right)^2\right]^{1/2}$$

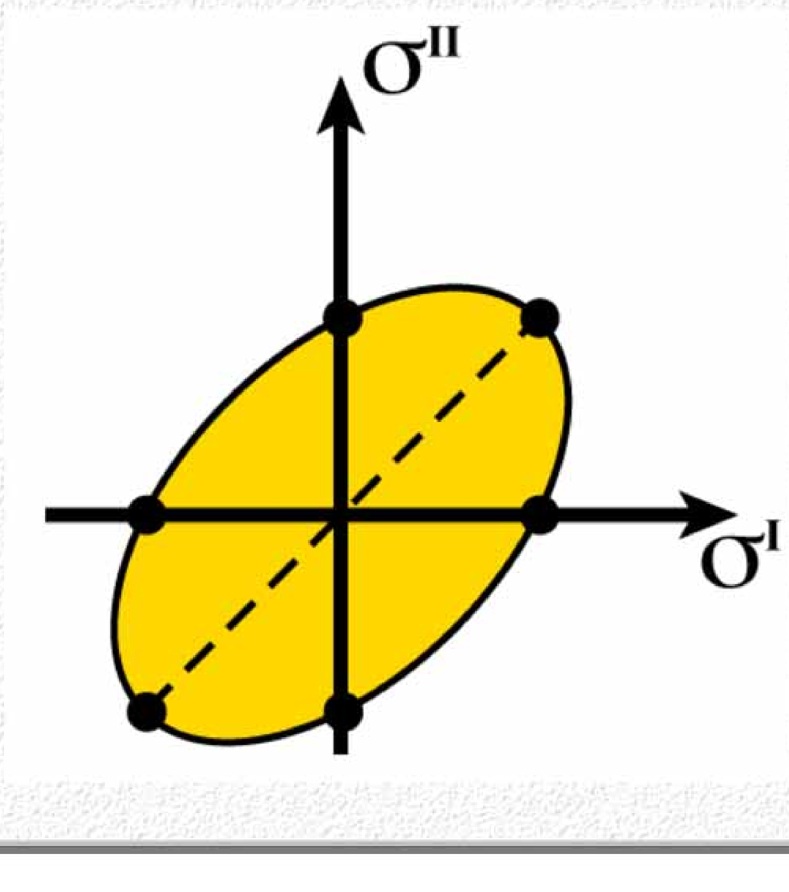
= effective (or von-Mises) stress







σ^{II}
1
2
5
3
σ_y
σ^{I}
0
σ_y
$-\sigma_y$
σ_y

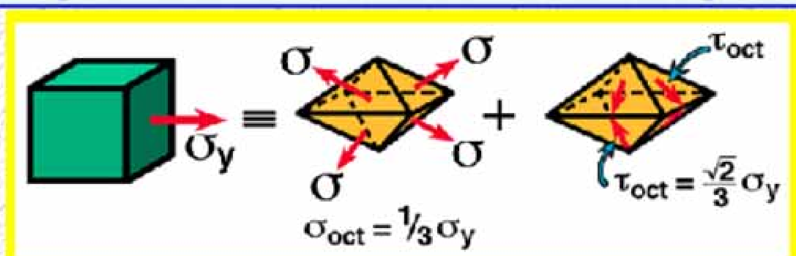


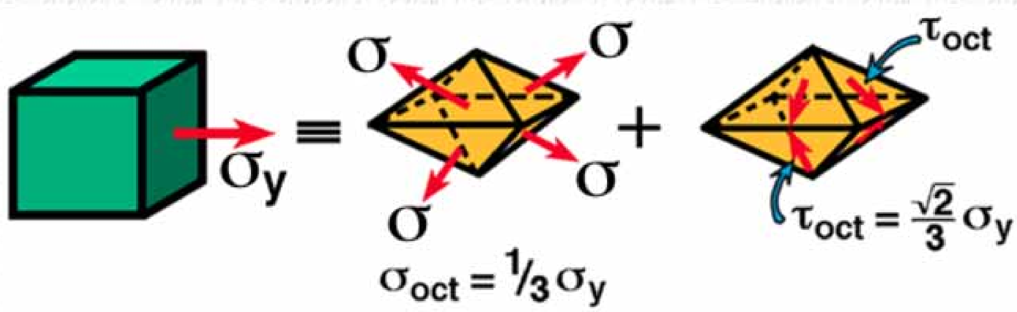
σ^{II}
σ^{I}

Maximum Octahedral Shear Stress Theory

- Yielding is assumed to begin at a point when the shearing stresses on the octahedral planes at the point reach a value equal to the octahedral shearing stress in uniaxial tension (or compression).
- In terms of principal stresses

$$\left(\tau_{oct.}\right)^2 = \frac{1}{9}\left[\left(\sigma^{I}-\sigma^{II}\right)^2+\left(\sigma^{II}-\sigma^{III}\right)^2+\left(\sigma^{III}-\sigma^{I}\right)^2\right] = \frac{2}{3}\frac{E}{1+\upsilon}U_{dist.}$$





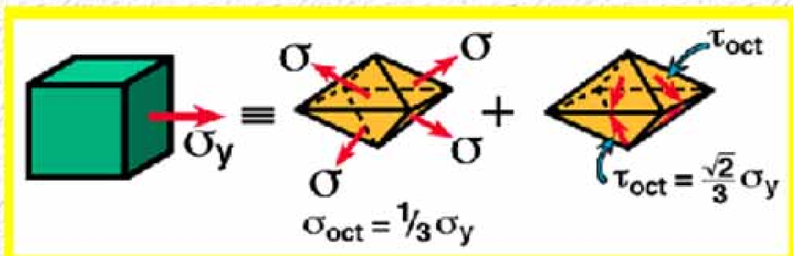
Maximum Octahedral Shear Stress Theory

- In terms of principal stresses

$$\boxed{\left(\tau_{oct.}\right)^2 = \frac{1}{9}\left[\left(\sigma^{I}-\sigma^{II}\right)^2+\left(\sigma^{II}-\sigma^{III}\right)^2+\left(\sigma^{III}-\sigma^{I}\right)^2\right] = \frac{2}{3}\frac{E}{1+\upsilon}U_{dist.}}$$

- Yielding begins when

$$\boxed{\tau_{oct.} = \frac{\sqrt{2}}{3}\,\sigma_{yield}}$$



Maximum Octahedral Shear Stress Theory

- Two-dimensional case, $\sigma^{III} = 0$

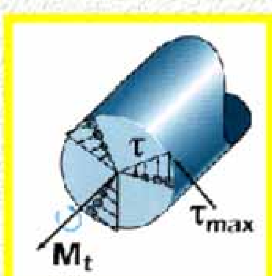
$$\boxed{\left(\sigma^{I}\right)^2 - \sigma^{I}\sigma^{II} + \left(\sigma^{II}\right)^2 = \left(\sigma_{yield}\right)^2}$$

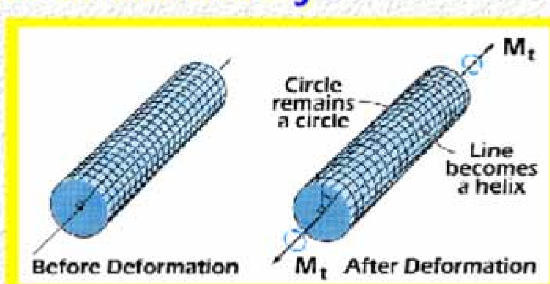
same as the maximum energy of distortion theory

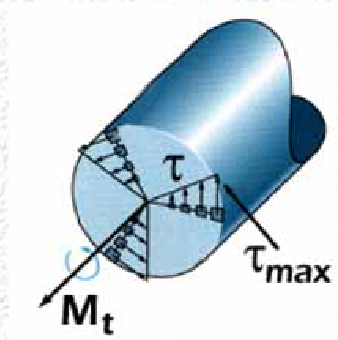
- For the torsion test
  - Maximum shear stress theory

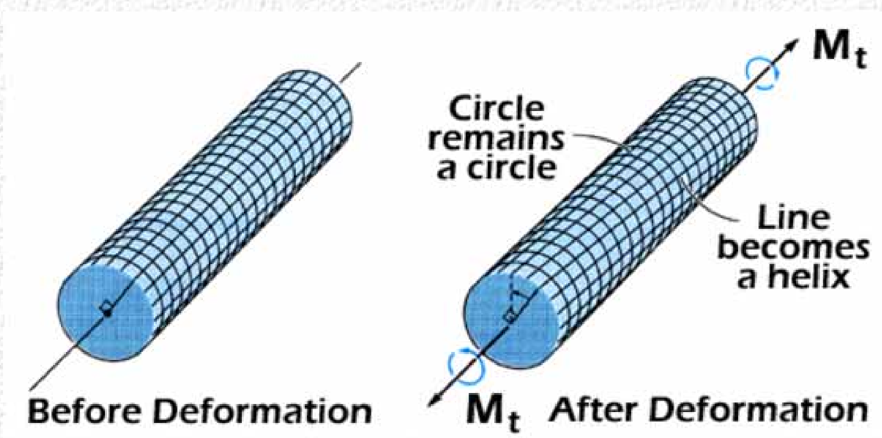
$$\boxed{\tau_{max} = \frac{M_t r}{I_p}}$$

$$\boxed{= \tau_{yield}}$$






τ
τmax
Mt


Mt
Circle remains a circle
Line becomes a helix
Before Deformation
Mt
After Deformation

Maximum Octahedral Shear Stress Theory

- Maximum energy of distortion

$$U_{dist.} = \frac{1}{2G}\left(\tau_{yield}\right)^2$$

$$= \frac{\left(1+\upsilon\right)}{E}\left(\tau_{yield}\right)^2$$

- Maximum octahedral shear stress

$$\tau_{oct.} = \sqrt{\frac{2}{3}}\ \tau_{yield}$$

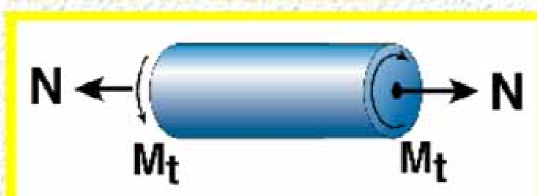
Shaft Subjected to Combined Axial Force and Twisting Moment

- Normal Stresses

$$\sigma = \frac{N}{A},\ A = \frac{\pi d^2}{4}$$

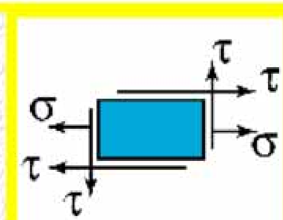
uniformly distributed over the cross section.

- Torsional Shear Stresses

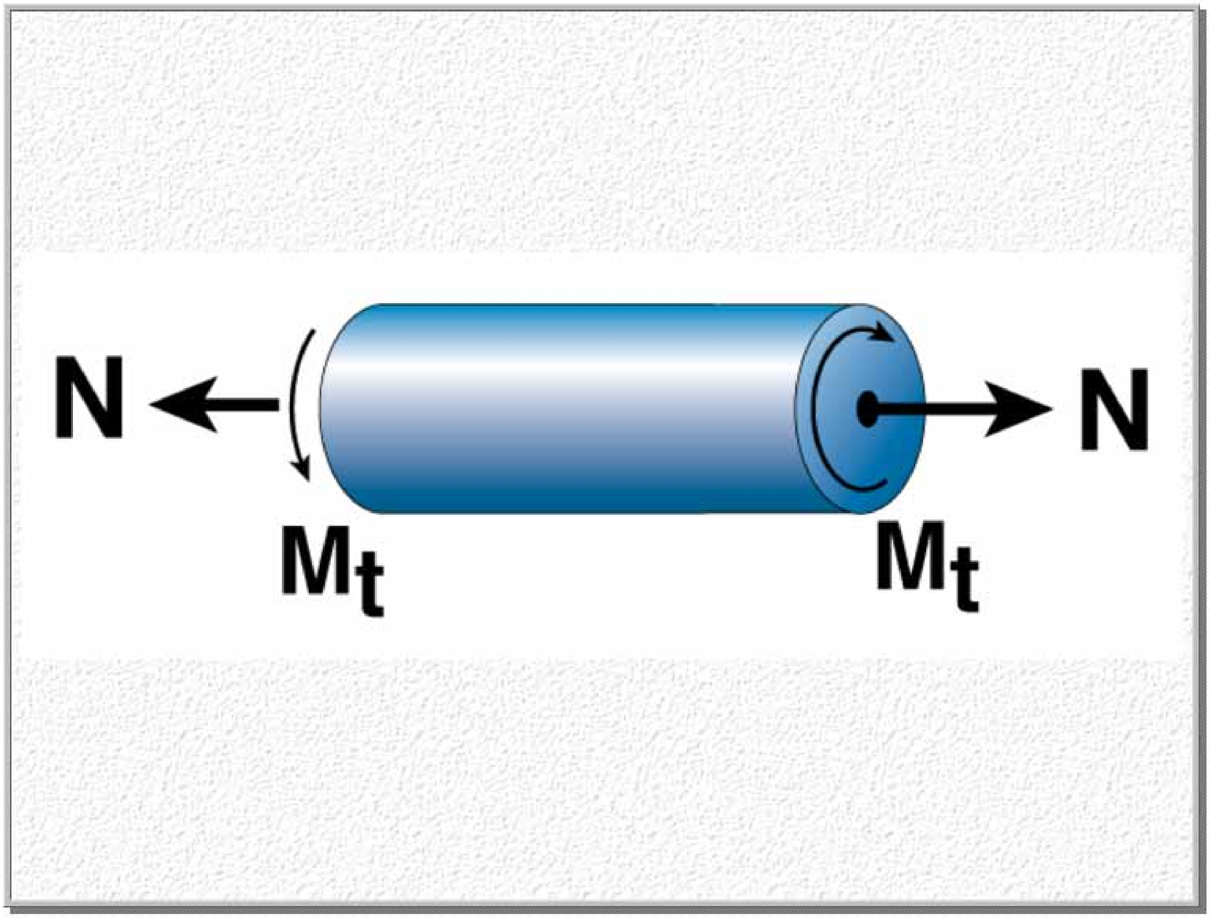


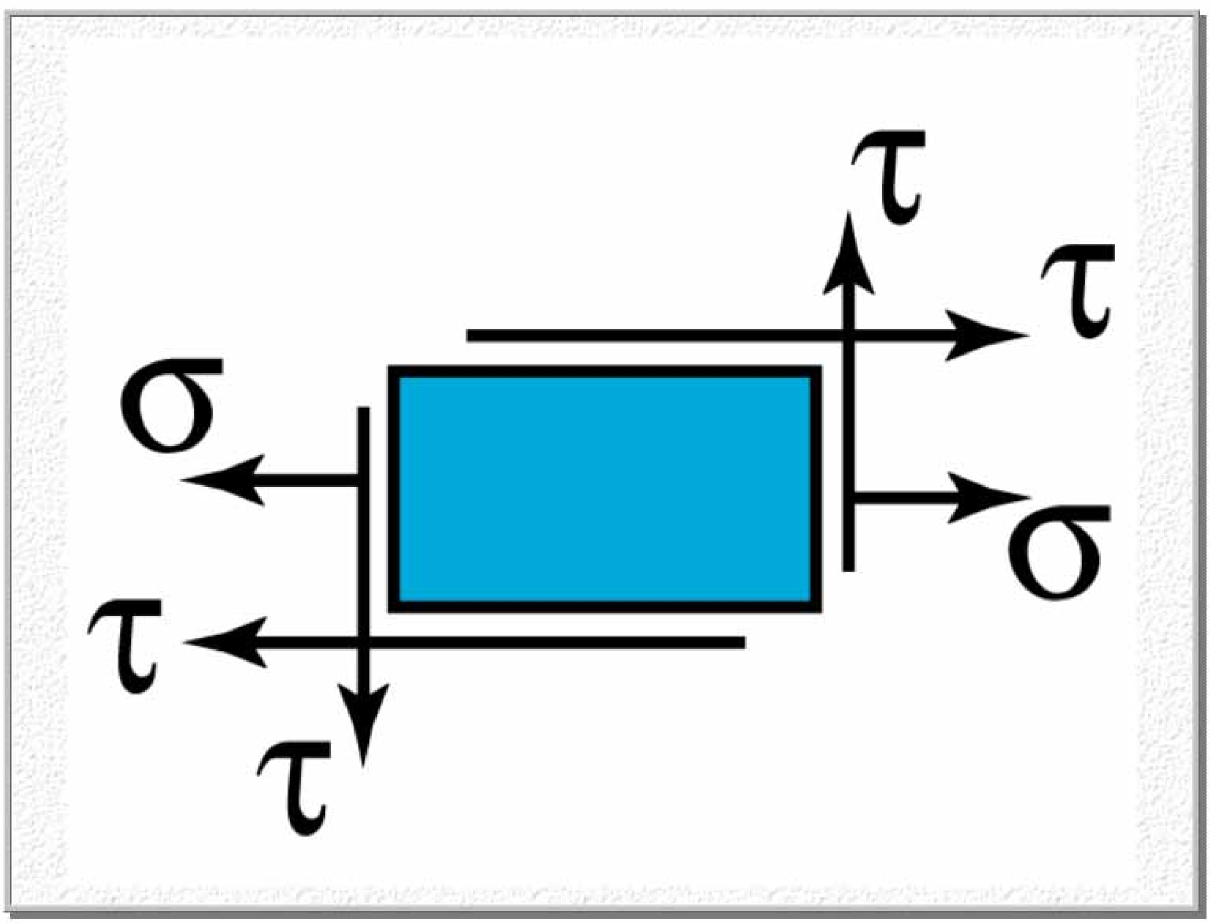
$$\tau_{max} = \frac{M_t r}{I_p},$$

$$\tau_{max} = \frac{16\,M_t}{\pi d^3}$$



Shear stresses are maximum at the perimeter of the cross section.


N
N
M_t
M_t


τ
τ
σ
σ
τ
τ

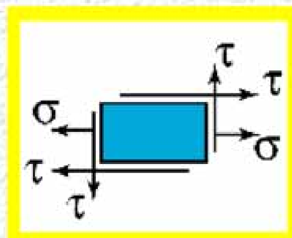
Shaft Subjected to Combined Axial Force and Twisting Moment

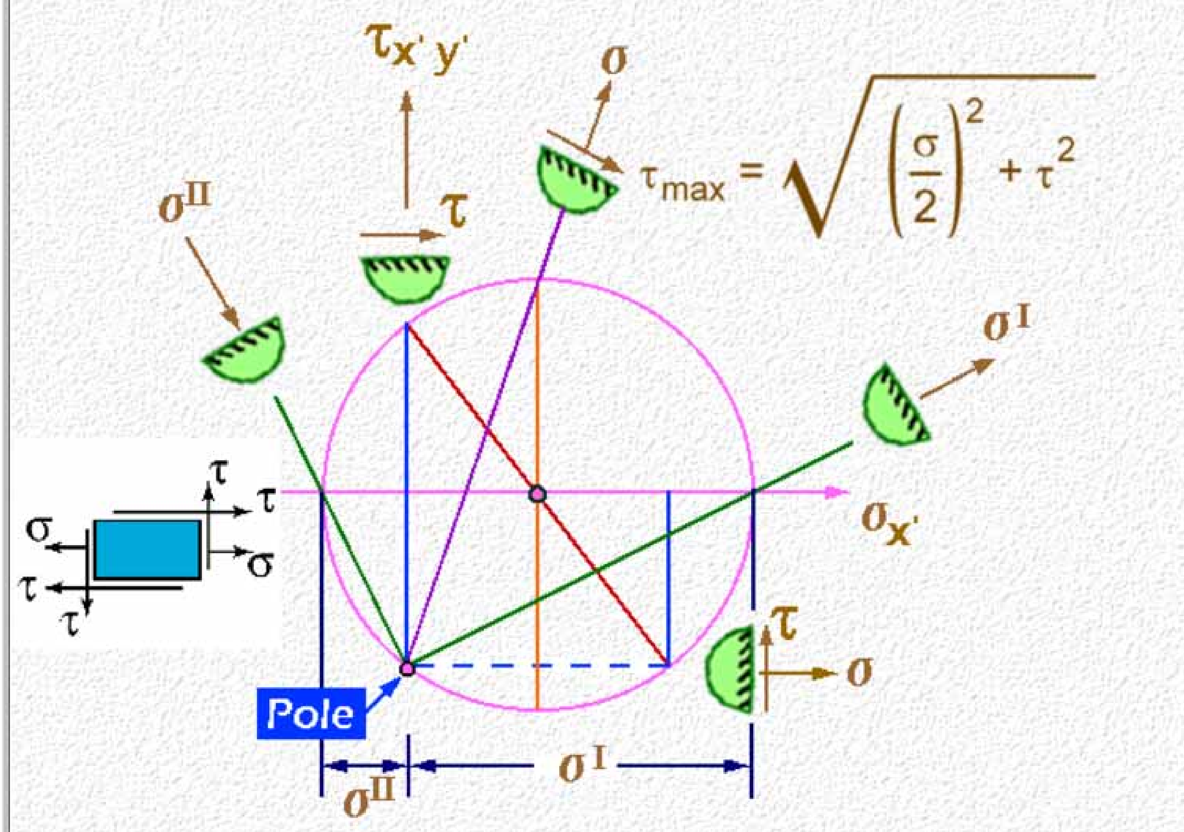
- <u>Maximum Principal Stresses</u>

  Occur at the perimeter of the cross section.

<u>Stress Matrix</u>

$$\left[\sigma\right] = \begin{bmatrix} \sigma & \tau & . \\ \tau & . & . \\ . & . & . \end{bmatrix}$$





Shaft Subjected to Combined Axial Force and Twisting Moment

- Principal Stresses

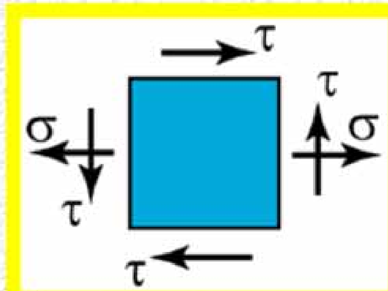
$$\begin{Bmatrix} \sigma^{I} \\ \sigma^{II} \end{Bmatrix} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

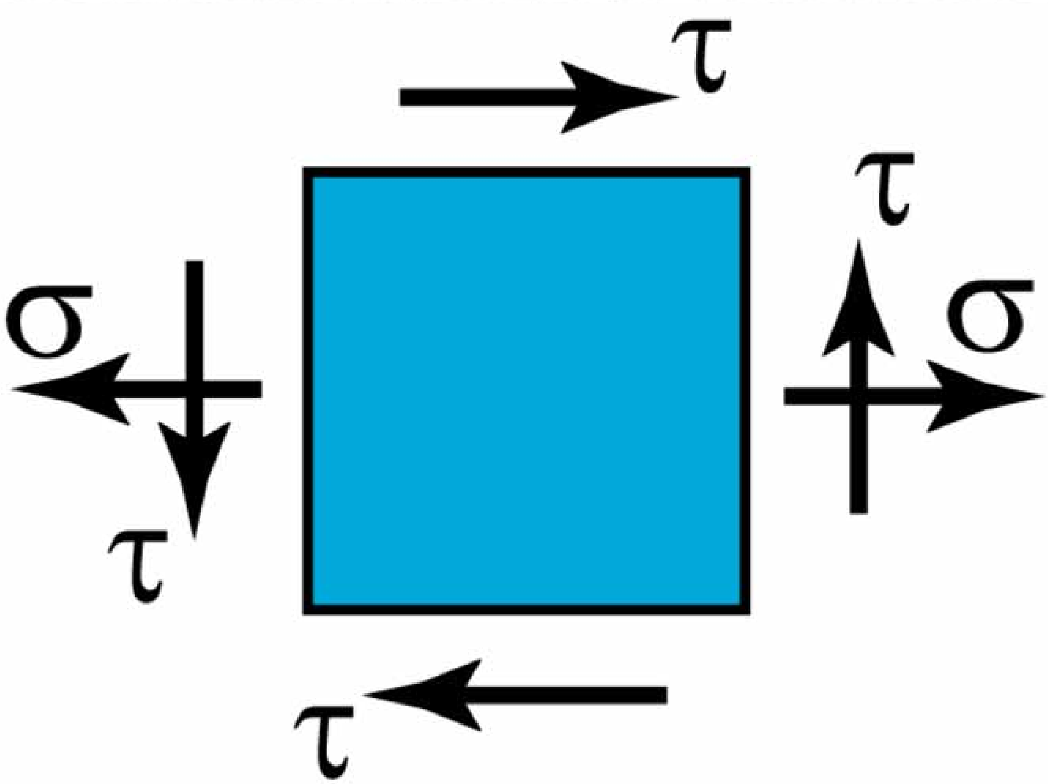
$$\sigma^{III} = 0$$

- Maximum Shear Stress

$$\tau_{max} = \pm \frac{1}{2}\left(\sigma^{I} - \sigma^{II}\right)$$

$$= \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$





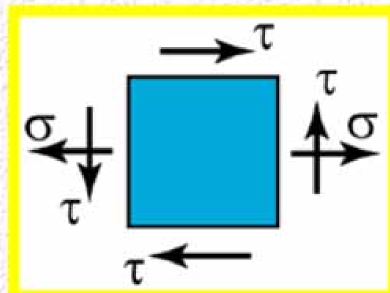
Shaft Subjected to Combined Axial Force and Twisting Moment

uniaxial stress state

$$\tau_{max} = \frac{1}{2}\sigma_{yield}$$

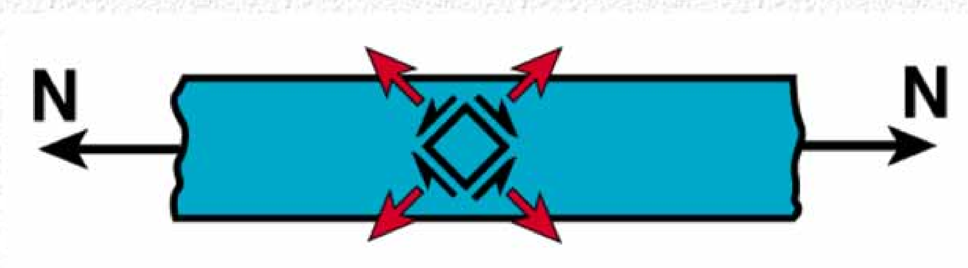
Therefore,

$$\sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{1}{2}\sigma_{yield}$$



or,

$$\sigma^2 + 4\tau^2 = \left(\sigma_{yield}\right)^2$$



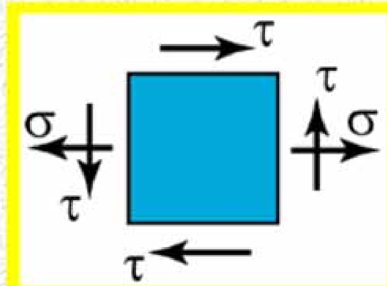
Shaft Subjected to Combined Axial Force and Twisting Moment

- Volumetric and Deviatoric Stress Components

$$\begin{bmatrix} \sigma & \tau & \cdot \\ \tau & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \frac{\sigma}{3} & \cdot & \cdot \\ \cdot & \frac{\sigma}{3} & \cdot \\ \cdot & \cdot & \frac{\sigma}{3} \end{bmatrix} + \begin{bmatrix} \frac{2}{3}\sigma & \tau & \cdot \\ \tau & -\frac{\sigma}{3} & \cdot \\ \cdot & \cdot & -\frac{\sigma}{3} \end{bmatrix}$$

- Strain Energy Densities

$$U_{tot} = \frac{\sigma^2}{2E} + \frac{\tau^2}{2G}$$



Shaft Subjected to Combined Axial Force and Twisting Moment

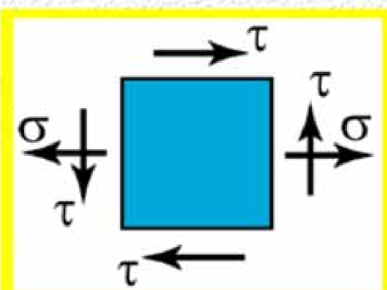
- Strain Energy Densities

$$U_{tot} = \frac{\sigma^2}{2E} + \frac{\tau^2}{2G}$$

$$U_{vol} = \tfrac{3}{2}\,\sigma_{vol} \times \varepsilon_{vol}$$

$$= \frac{3}{2}\,\frac{\sigma}{3} \cdot \left(\frac{(1-2\upsilon)}{E}\,\frac{\sigma}{3} \right)$$

$$= \left(\frac{(1-2\upsilon)}{6E}\,\sigma^2 \right)$$

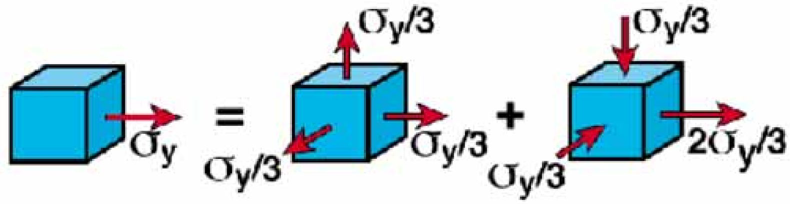


$$U_{dist} = U_{tot} - U_{vol}$$

$$= \frac{1+\upsilon}{3E}\left(\sigma^2 + 3\tau^2\right)$$

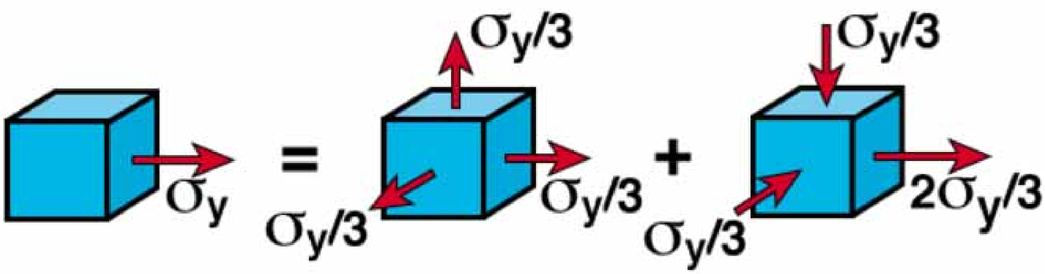
Shaft Subjected to Combined Axial Force and Twisting Moment

For uniaxial stress state

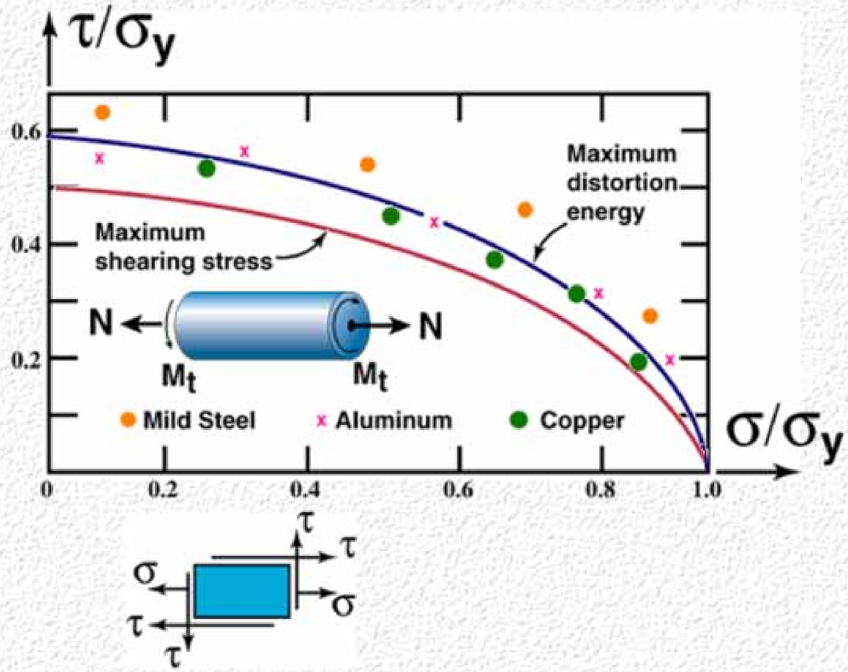


$$U_{dist} = \frac{1+\upsilon}{3E}\left(\sigma_{yield}\right)^2$$

$$\sigma^2 + 3\,\tau^2 = \left(\sigma_{yield}\right)^2$$



Shaft Subjected to Combined Axial Force and Twisting Moment



Application of Failure Theories to Design

- A margin of safety is introduced in the design.
- Factor of safety, f, is the ratio of the failure load to the design load.
- Stresses are assumed to be proportional to the loads. Therefore,

$$\boxed{f = \frac{\text{failure stress}}{\text{design stress}}}$$

Application of Failure Theories to Design

- Maximum principal stress theory

$$\sigma_{design} = \frac{1}{f}\sigma_{yield}$$

- Maximum strain theory

$$\varepsilon_{design} = \frac{1}{f}\frac{\sigma_{yield}}{E}$$

Application of Failure Theories to Design

- Maximum shear stress theory

Tension test

$$\tau_{design} = \frac{1}{2f}\sigma_{yield}$$

Torsion test

$$\tau_{design} = \frac{1}{f}\sigma_{yield}$$

Application of Failure Theories to Design

- Maximum total strain energy theory

$$U_{design} = \frac{1}{2E}\left(\frac{\sigma_{yield}}{f}\right)^2$$

Application of Failure Theories to Design

- Maximum energy of distortion theory

Tension test

$$U_{dist.}\Big|_{design} = \frac{1}{6G}\left(\frac{\sigma_{yield}}{f}\right)^2$$

Torsion test

$$U_{dist.}\Big|_{design} = \frac{1}{2G}\left(\frac{\tau_{yield}}{f}\right)^2$$

Application of Failure Theories to Design

- **Maximum octahedral shear stress theory**
  Tension test

$$\tau_{oct.}\Big|_{design} = \frac{\sqrt{2}}{3}\,\frac{\sigma_{yield}}{f}$$

Torsion test

$$\tau_{oct.}\Big|_{design} = \sqrt{\frac{2}{3}}\,\frac{\tau_{yield}}{f}$$